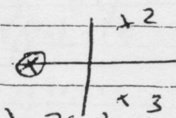
2009.

1 a) $f(x,y) = 3xy^2 - 2x^2 - 3y^2 - 8x + 2$

$f_x = 0 \Rightarrow 3y^2 - 4x - 8 = 0$

$f_y = 0 \Rightarrow 6xy - 6y = 0 \Rightarrow y = 0$ or $x = 1$ $\Big\} \Rightarrow x = -2$ if $y = 0$, or $y = \pm 2$ if ...

$\Rightarrow$ Critical points at $(-2,0)^{(1)}$, $(1,2)^{(2)}$, $(1,-2)^{(3)}$,



$f_{xx} = -4$, $f_{yy} = 6(x-1)$, $f_{xy} = 6y$, $\Delta = f_{xx}f_{yy} - f_{xy}^2 = 24(1-x) - 36y^2$

1) $\Delta = 24.3, > 0$, & $f_{xx} < 0 \Rightarrow (-2,0)$ is a maximum

2 & 3), as $x = 1$, $\Delta < 0 \Rightarrow (1,2)$ & $(1,-2)$ are saddle point.

b) Points will be the solutions of the equations $f_x = f_y = f_z = 0$ & the two constraints $\underline{x^2 + y^2 = z^2}$ ① & $\underline{x + y + 2 = z}$ ②. - The method of Lagrange multipliers.

$f = x^2 + y^2 + z^2 - \lambda(x^2 + y^2 - z^2) - \mu(x + y - z)$ (3 variables with 2 constraints, mentioned but no examples

$f_x = 0 \Rightarrow 2x - \lambda 2x - \mu = 0 \Rightarrow \underline{2x(1-\lambda) = \mu}$.

$f_y = 0 \Rightarrow 2y - \lambda 2y - \mu = 0 \Rightarrow \underline{2y(1-\lambda) = \mu}$

$f_z = 0 \Rightarrow 2z + 2\lambda z \mp \mu = 0 \Rightarrow \underline{2z(1+\lambda) = -\mu}$

① $\Rightarrow 2 \cdot \frac{1}{4} \cdot \mu^2/(1-\lambda)^2 = \frac{1}{4}\mu^2/(1+\lambda)^2 \Rightarrow \underline{1 - \lambda = \pm\sqrt{2}(1+\lambda)}$ ie $\frac{1}{1+\lambda} = \frac{\pm\sqrt{2}}{1-\lambda}$

② $\Rightarrow 2 \cdot (\mu/2(1-\lambda)) + \mu/2(1+\lambda) = -2 \Rightarrow \mu/1-\lambda \left[1 \pm \sqrt{2}/2\right] = -2 \Rightarrow$

$\mu/1-\lambda = -2 \cdot 2/2 \pm \sqrt{2} = -4(2 \mp \sqrt{2})/2 = -2(2 \mp \sqrt{2})$

& so $x = y = \mu/2(1-\lambda) = \underline{\underline{-2 \pm \sqrt{2}}}$, $z = -\mu/2(1+\lambda) = \mp\frac{\sqrt{2}}{2}\frac{\mu}{1-\lambda}$

$= \mp\sqrt{2}(-1)(2 \mp \sqrt{2}) = \underline{\underline{\pm 2\sqrt{2} - 2}}$

2 a) EL eqns are $\partial F/\partial y - d/dx(\partial F/\partial y') = 0^{(*)}$ $\quad y(x_1) = y_1,\ y(x_2) = y_2$

If $\partial F/\partial x = 0$, then considering $\frac{d}{dx}(F - y'\partial F/\partial y') = \partial F/\partial y \cdot y' + \partial F/\partial y'\, y'' - y''\partial F/\partial y' - y'\frac{d}{dx}(\partial F/\partial y') = y'(\partial F/\partial y - d/dx(\partial F/\partial y')) = 0$. Hence

$$F - y'\partial F/\partial y' = \text{const}$$

b) $A[y] = \frac{1}{2}\int_0^T y'^2 - \omega^2 y^2\,dx$, $y(0) = a$, $y(T) = 0$

EL eqns with $F = \frac{1}{2}(y'^2 - \omega^2 y^2)$ give $-\omega^2 y - \frac{d}{dx}y' = 0 \Rightarrow y'' + \omega^2 y = 0$

$y(0) = a$, $y(T) = 0$

Solution is $\quad y_e = A\cos\omega x + B\sin\omega x$

Integration by parts yeilds

$A[y] = \left[\frac{1}{2}y'y\right]_0^T - \frac{1}{2}\int yy'' + \omega^2 y^2\,dx$. But $yy'' + \omega^2 y^2 = y(y'' + \omega^2 y) = 0$ for the extremal curve.

$$\Rightarrow A[y_e] = \frac{1}{2}\left[y_e' y_e\right]_0^T = \frac{1}{2}\big(y_e'(T)\underbrace{y_e(T)}_{0} - \underbrace{y_e(0)}_{a}y_e'(0)\big) = -\frac{1}{2}a\,y_e'(0)$$

$= -\frac{1}{2}a\omega B$ in the solution above

$y_e(0) = a \Rightarrow A = a$ & $y_e(T) = 0 \Rightarrow 0 = a\cos\omega T + B\sin\omega T \Rightarrow B = -a\cot\omega T$

$$\Rightarrow A[y_e] = \frac{1}{2}a^2\omega\cot(\omega T)$$

3 a) $z_x + z_y = z^2$, χ eqns $dx/dt = 1$, $dy/dt = 1$, $dz/dt = z^2$. Write condition $z(x,0) = f(x)$ as $x = s, y = 0, z = f(s)$ & solve χ eqns with these conditions $\Rightarrow$ $\underline{x = t+s}$, $\underline{y = t}$, $\int dz/z^2 = \int dt \Rightarrow t = 1/f(s) - 1/z$ & $\underline{z = f(s)/1 - tf(s)}$. Eliminating s & t in favour of x & y gives $\Rightarrow t = y$, $s = x - y$ & $\underline{z(x,y) = f(x-y)/(1 - yf(x-y))}$.

b) $yz_x - 2xyz_y = 2xz$: χ eqns are $dx/y = dy/-2xy = dz/2xz$, So that $-2xy\,dx = y\,dy \Rightarrow d(y + x^2) = 0$, $y + x^2 = \eta$ a const on χ.
Change variables from x, y to x & η; $y(z_x + 2x\cancel{z_\eta}) - 2xy\cancel{z_\eta} = 2xz$
$\Rightarrow \left.dz/dx\right|_\eta = \dfrac{2xz}{\eta - x^2} \Rightarrow \int dz/z = \int \dfrac{2x}{\eta - x^2} \Rightarrow \ln z = -\ln(\eta - x^2) + f(\eta$
& $\underline{z(x,y) = f(y + x^2)/y}$.

c) $xzz_x - yzz_y = x^2 - y^2$; χ eqns are $dx/xz = \dfrac{dy}{-yz} = \dfrac{dz}{x^2 - y^2}$
So $dx/x + dy/y = 0 \Rightarrow xy = \text{Const}$
$x\,dx + y\,dy = (x^2 - y^2)z\,dz/x^2 - y^2 \Rightarrow x\,dx + y\,dy - z\,dz = 0$, $x^2 + y^2 - z^2 = \text{Co}$
Lagranges method then gives solution as $x^2 + y^2 - z^2 = f(xy)$

4) a) $c^2 z_{xx} - z_{tt} = 0$ Switch to coordinates $\xi = x+ct$ & $\eta = x-ct$,

$\partial_x = 1\partial_\xi + 1\partial_\eta$, $\partial_{xx} = \partial_{\xi\xi} + 2\partial_{\xi\eta} + \partial_{\eta\eta}$

$\partial_t = c\partial_\xi - c\partial_\eta$, $\partial_{tt} = c^2\partial_{\xi\xi} - 2c^2\partial_{\xi\eta} + c^2\partial_{\eta\eta}$

$c^2 z_{xx} - z_{tt} = 0 \Rightarrow 4c^2 z_{\xi\eta} = 0 \Rightarrow z_{\xi\eta} = 0$

$\Rightarrow z_\xi = f'(\xi)$, $z = f(\xi) + g(\eta) = f(x+ct) + g(x-ct)$

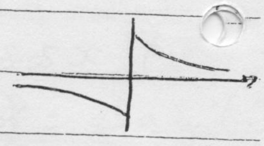
f & g need to be chosen so $z(x,0) = F(x)$, $z_t(x,0) = G(x)$

$\Rightarrow f + g = F$ & $c(f' - g') = G$ ie $f - g = \frac{1}{c}\int_\alpha^x G(s)ds$

$\Rightarrow f = \frac{1}{2}F + \frac{1}{2c}\int_\alpha^x G(s)ds$, $g = \frac{1}{2}F - \frac{1}{2c}\int_\alpha^x G(s)ds$

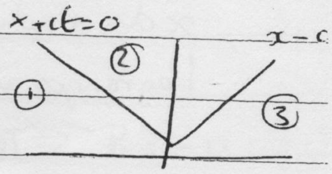
$$z(x,t) = \frac{1}{2}\left(F(x+ct) + F(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds$$

b) $z(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds$, $G(s) = -e^s \quad s<0$
$\qquad = e^{-s} \quad s>0$



So In ①, $z = \frac{1}{2c}\int_{x-ct}^{x+ct} -e^s ds = \frac{1}{2c}\left(e^{x-ct} - e^{x+ct}\right)$

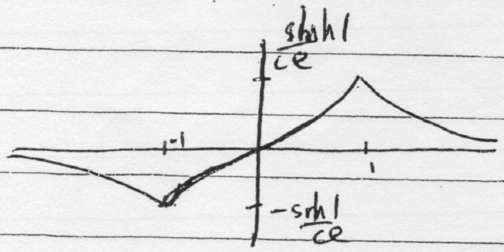
$= -e^x \sinh(ct)/c$, $x < -ct$



In ②, $z = \frac{1}{2c}\left\{\int_{x-ct}^{0} -e^s ds + \int_0^{x+ct} e^{-s}ds\right\} = \frac{1}{2c}\left\{e^{x-ct} - 1 + 1 - e^{-x-ct}\right\}$

$= e^{-ct}\sinh(x)/c$, $-ct < x < ct$

In ③, $z = \frac{1}{2c}\int_{x-ct}^{x+ct} e^{-s}ds = \frac{1}{2c}\left\{e^{-x+ct} - e^{-x-ct}\right\} = e^{-x}\sinh(ct)/c$



Max & Min values are $\pm$ sinh1/ce

5) $\alpha^2\theta_{xx} = \theta_t$, $\theta_x(-L) = \theta_x(L) = 0$ & at $t=0$

$\theta = X(x)T(t) \Rightarrow \alpha^2 X''/X = T'/T = \text{Const}$. To satisfy b.cs we require Const $= 0$ or $-$ve. Write it as $-\alpha^2 p^2$. If zero then $X = Ax + B$ & bc require $A = 0$, $B = A_0$ say. If nonzero then $X = A\cos px + B\sin px$. The initial conditions are even so $B = 0$ & bc require $\sin pL = 0 \Rightarrow p = n\pi/L$ $n = 1, 2, \dots$ Hence $\theta = A_0 + \sum_1^\infty A_n \cos(n\pi x/L)\, e^{-n^2\pi^2\alpha^2 t/L^2}$ as $T'/T = -\alpha^2 p^2 = -\alpha^2 n^2\pi^2/L^2$ & $T \propto e^{-n^2\pi^2\alpha^2 t/L^2}$

Coefficients require $\theta(x,0) = A_0 + \sum_1^\infty A_n \cos(n\pi x/L)$

$\Rightarrow 2LA_0 = \int_{-L}^{L} \theta(x,0)\,dx = 2a \Rightarrow A_0 = a/L$

$2LA_n/2 = \int_{-L}^{L} \theta(x,0)\cos\frac{n\pi x}{L}\,dx = 2\int_0^a \cos\frac{n\pi x}{L}\,dx = \frac{2L}{n\pi}\sin\frac{n\pi a}{L}$

$\Rightarrow A_n = \frac{2}{n\pi}\sin n\pi a/L$

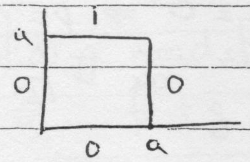
$$\theta(x,t) = \frac{a}{L} + 2\sum_1^\infty \frac{\sin(n\pi a/L)}{n\pi}\cos\frac{n\pi x}{L}\, e^{-n^2\pi^2\alpha^2 t/L^2}$$

As $a \to 0$ $\sin(n\pi a/L) \to n\pi a/L$ & $L\theta(x,t)/a \approx 1 + 2\sum \cos\frac{n\pi x}{L}\, e^{-n^2\pi^2\alpha^2 t/L^2}$

$t > 0$ for convergence.

6. a) $\nabla^2 u = 0$. If $u = X(x)Y(y)$, $X''/X + Y''/Y = 0 \Rightarrow X''/X = \alpha^2$, Y

say, giving exponential behaviour in x & oscillatory in y. Alternatively if $X''/X = -\beta^2$ & $Y''/Y = \beta^2$, the roles are reversed. If $X''/X = Y''/Y = 0$ then both X & Y are linear in x & y.



Homogenous bc in $x \Rightarrow$ oscillatory in x, $\sin\beta x$, $\cos\beta x$, $\sinh\beta y$, $\cosh\beta y$; $u(0,y)=0 \Rightarrow \sin\beta x$ only; $u(x,0)=0$ $\Rightarrow \sinh\beta y$ only. Finally $u(a,y)=0 \Rightarrow \sin\beta a = 0$ & so $\beta = n\pi/a$, $n=1,2..$

Hence $u(x,y) = \sum_1^\infty A_n \sin(n\pi x/a)\sinh(n\pi y/a)$

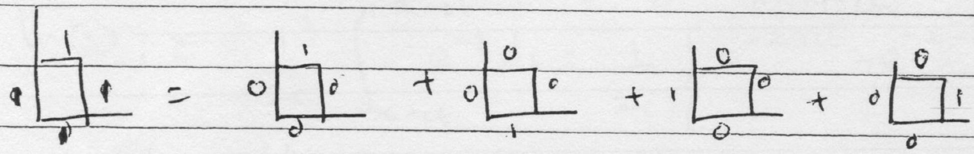
A_n are obtained from requiring $u(x,a)=1 \Rightarrow 1 = \sum_1^\infty A_n \sinh(n\pi)\sin(n\pi x/a)$

$\Rightarrow \frac{a}{2} A_n \sinh(n\pi) = \int_0^a \sin\frac{n\pi x}{a}\,dx = \frac{a}{n\pi}(1-\cos n\pi) = \frac{a}{n\pi}(1-(-1)^n)$

$= 0$ for even n & $2a/(n\pi)$ for odd n. Write $n = \lambda_m = 2m+1$

to get $A_m = 4/n\pi\sinh(n\pi) = 4/\lambda_m\pi\sinh(\lambda_m\pi)$.

$$u(x,y) = \frac{4}{\pi}\sum_1^\infty \sin(\lambda_m \pi x/a)\sinh(\lambda_m\pi y/a)/\lambda_m\sinh(\lambda_m\pi), \quad \lambda_m = 2m+1$$



$$\Rightarrow u(x,y) = \frac{4}{\pi}\sum_{m=1}^\infty \left\{ \frac{\sin(\lambda_m\pi x/a)}{\lambda_m\sinh(\lambda_m\pi)}\left(\sinh(\lambda_m\pi y/a) + \sinh(\lambda_m\pi(a-y)/a)\right) + \frac{\sinh(\lambda_m\pi x/a)}{\lambda_m\sinh(\lambda_m\pi)}\left(\sin(\lambda_m\pi y/a) + \sin(\lambda_m\pi(a-y)/a)\right)\right.$$

Solution is obviously $u=1$ & evaluating at centre of square gives

$$1 = 4/\pi\sum_1^\infty \frac{\sin(\lambda_m\pi/2)}{\lambda_m(\sin\lambda_m\pi)}\left(\sinh(\lambda_m\pi/2) + \sinh(\lambda_m\pi/2)\right) + \frac{\sinh\lambda_m\pi/2}{\lambda_m\sinh\lambda_m\pi}\left(\sin\lambda_m\pi/2 + \sin\lambda_m\pi/2\right) = \frac{4}{\pi}.4.\sum_1^\infty \frac{\sin(\lambda_m\pi/2)\cdot\cancel{\sinh\lambda_m\pi/2}}{\lambda_m\,2.\sinh\cancel{\lambda_m\pi/2}\cosh\lambda_m\pi/2} \quad (7)$$

but $\sin\lambda_m\pi/2 = \sin(\pi/2 + m\pi) = (-1)^m \Rightarrow$ result